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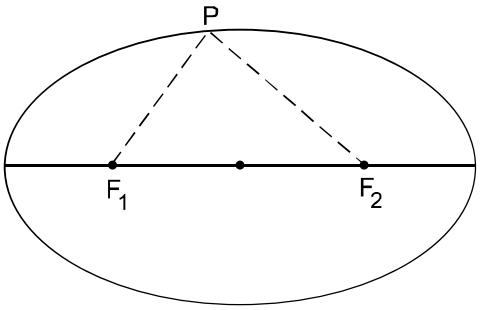
Ellipse

ELLIPSE *It is the path traced by a point which moves in a plane in such a way that the sum of its distance from two fixed points in the plane is a constant.*

The two fixed points are called the *foci* of the ellipse.

NOTE The plural of focus is foci.

In the given figure, F_1 and F_2 are two fixed points and P is a point which moves in such a way that $PF_1 + PF_2 = constant$.



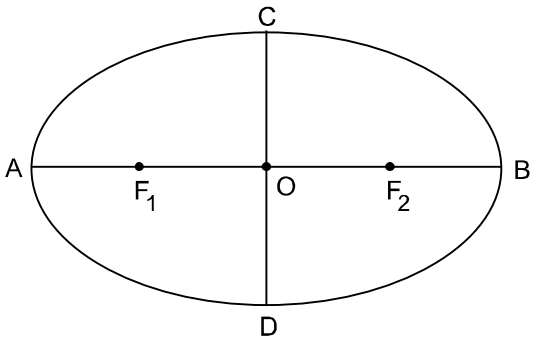
The path traced by the point P is called an ellipse, and the points F_1 and F_2 are called its *foci*.

SOME MORE TERMS RELATED TO AN ELLIPSE

(I) CENTRE OF THE ELLIPSE

The midpoint of the line segment joining the foci, is called the centre of the ellipse.

In the given figure, F_1 and F_2 are the foci of the ellipse and O is its *centre*, where $OF_1 = OF_2$.



(II) AXES OF THE ELLIPSE

MAJOR AXIS: *The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.*

In the given figure, AB is the major axis of the ellipse.

MINOR AXIS: *The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.*

In the given figure, CD is the minor axis of the ellipse.

(III) VERTICES OF AN ELLIPSE

The end points of the major axis of an ellipse are called its vertices.

In the given figure, A and B are the vertices of the ellipse.

AN IMPORTANT NOTE In an ellipse, we take:

Length of the major axis = $AB = 2a$.

Length of the minor axis = $CD = 2b$.

Distance between the foci = $F_1F_2 = 2c$.

Length of the semi-major axis = a.

Length of the semi-minor axis = b.

(IV) ECCENTRICITY OF AN ELLIPSE

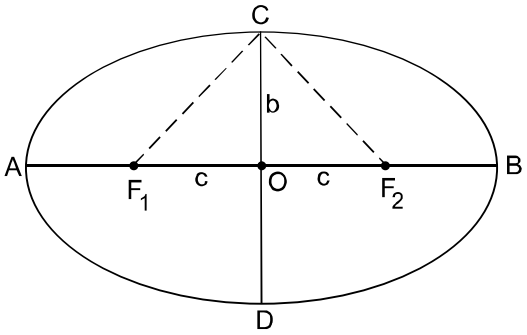
The ratio $\frac{c}{a}$ is always constant and it is denoted by e, called the *eccentricity* of the ellipse.

For an ellipse, we have $0 < e < 1$ $\quad\left[\because\ c < a \ \Leftrightarrow\ e = \frac{c}{a} < 1\right]$.

AN IMPORTANT RESULT

In the given ellipse, it is being given that $AB = 2a$, $CD = 2b$ and $F_1F_2 = 2c$.

Prove that: $(a^2 - c^2) = b^2$.



PROOF Clearly, COD is the perpendicular bisector of F_1F_2.

Join F_1C and F_2C.

Now, $F_1O = c$, $OC = b$ $\Rightarrow$ $F_1C = \sqrt{c^2 + b^2}$.

And, $OF_2 = c$, $OC = b$ $\Rightarrow$ $F_2C = \sqrt{c^2 + b^2}$.

Since B and C both lie on the given ellipse, we have

$F_1B + F_2B = F_1C + F_2C =$ constant [by the definition of an ellipse]

$\Rightarrow$ $(F_1O + OB) + (OB - OF_2) = F_1C + F_2C$

$\Rightarrow$ $(c + a) + (a - c) = 2\sqrt{c^2 + b^2}$

$[\because\ F_1O = c,\ OF_2 = c,\ OB = a$ and $F_1C = F_2C = \sqrt{c^2 + b^2}]$

$\Rightarrow$ $a = \sqrt{c^2 + b^2}$ $\Rightarrow$ $a^2 = (c^2 + b^2)$ $\Rightarrow$ $(a^2 - c^2) = b^2$.

Hence, $(a^2 - c^2) = b^2$.

STANDARD EQUATION OF AN ELLIPSE

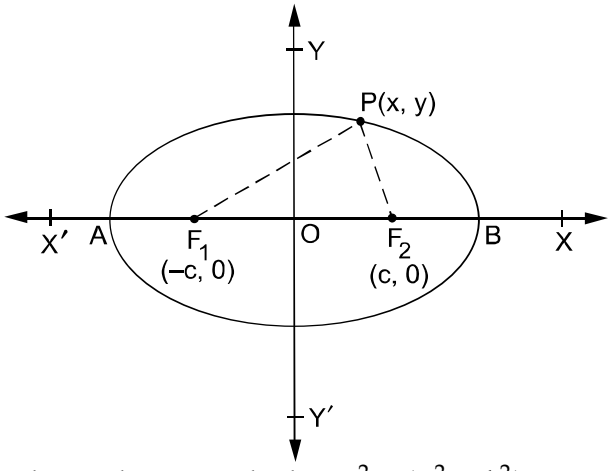
THEOREM *Prove that the standard equation of an ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the lengths of the semi-major axis and the semi-minor axis respectively and $a > b$.

PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let us consider an ellipse in which a and b are the lengths of the semi-major axis and semi-minor axis respectively.



Choose a real number c such that $c^2 = (a^2 - b^2)$.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the two fixed points and let $P(x, y)$ be an arbitrary point on the given ellipse, moving on it in such a way that $PF_1 + PF_2 = 2a$. Then,

$$PF_1 + PF_2 = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

[on squaring both sides]

$$\Rightarrow (x+c)^2 - (x-c)^2 - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = a - \frac{cx}{a}$$ [on dividing both sides by $-4a$]

$$\Rightarrow (x-c)^2 + y^2 = a^2 - 2cx + \frac{c^2x^2}{a^2}$$ [on squaring both sides]

$$\Rightarrow x^2 - \frac{c^2x^2}{a^2} + y^2 = (a^2 - c^2)$$

$$\Rightarrow x^2\left(1 - \frac{c^2}{a^2}\right) + y^2 = (a^2 - c^2)$$

$$\Rightarrow \frac{x^2(a^2 - c^2)}{a^2} + y^2 = (a^2 - c^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{(a^2 - c^2)} = 1$$ [on dividing both sides by $(a^2 - c^2)$]

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ [$\because (a^2 - c^2) = b^2$].

Thus, every point on the ellipse satisfies the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Conversely, let the equation of the given curve be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P(x, y)$ be an arbitrary point on this curve.

Then, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)$

$$\Rightarrow y^2=\frac{b^2(a^2-x^2)}{a^2}. \quad \ldots \text{(i)}$$

Also, let $(a^2-b^2)=c^2$. … (ii)

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be two fixed points on the x-axis. Then,

$$PF_1=\sqrt{(x+c)^2+y^2}$$

$$=\sqrt{(x+c)^2+\frac{b^2(a^2-x^2)}{a^2}} \quad \text{[using (i)]}$$

$$=\sqrt{(x+c)^2+\frac{(a^2-c^2)(a^2-x^2)}{a^2}} \quad \text{[using (ii)]}$$

$$=\sqrt{a^2+2cx+\frac{c^2x^2}{a^2}}=\sqrt{\left(a+\frac{cx}{a}\right)^2}=\left(a+\frac{cx}{a}\right)$$

Similarly, $PF_2=\left(a-\frac{cx}{a}\right)$.

$\therefore \quad PF_1+PF_2=\left(a+\frac{cx}{a}+a-\frac{cx}{a}\right)$

$\Leftrightarrow \quad PF_1+PF_2=2a.$

This shows that the given curve is an ellipse.

Hence, the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

AN IMPORTANT NOTE The foci of an ellipse always lie on the major axis.

HORIZONTAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the larger denominator then its major axis lies along the x-axis.

Such an ellipse is called a *horizontal ellipse.*

Thus, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an horizontal ellipse, if $a^2>b^2$.

EXAMPLE $\frac{x^2}{16}+\frac{y^2}{9}=1$ is an horizontal ellipse.

LATUS RECTUM OF A HORIZONTAL ELLIPSE

The latus rectum of an ellipse is a line segment perpendicular to the major axis, passing through any of the foci with end points lying on the ellipse.

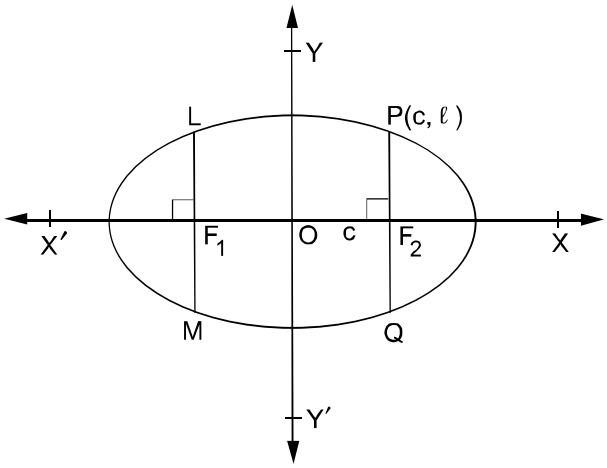
To Find the Length of Latus Rectum of the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b.$

Let $X'OX$ and YOY' be the coordinate axes.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the given ellipse, where $c^2 = (a^2 - b^2)$.

Let LF_1M and PF_2Q be the latus rectums.

Let $PF_2 = l$. Then, the coordinates of P are $P(c, l)$.



Since $P(c, l)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$\frac{c^2}{a^2} + \frac{l^2}{b^2} = 1 \Rightarrow \frac{l^2}{b^2} = \left(1 - \frac{c^2}{a^2}\right)$$

$$\Rightarrow l^2 = b^2\left(1 - \frac{c^2}{a^2}\right) = b^2\left[1 - \frac{(a^2 - b^2)}{a^2}\right] \qquad [\because\ c^2 = (a^2 - b^2)]$$

$$\Rightarrow l^2 = \frac{b^4}{a^2} \Rightarrow l = \frac{b^2}{a}.$$

$$\therefore\ PF_2 = l = \frac{b^2}{a} \Leftrightarrow PF_2Q = 2l = \frac{2b^2}{a}.$$

Similarly, $LF_1M = 2l = \frac{2b^2}{a}$.

Hence, the length of the latus rectum is $\frac{2b^2}{a}$.

RESULTS ON HORIZONTAL ELLIPSE

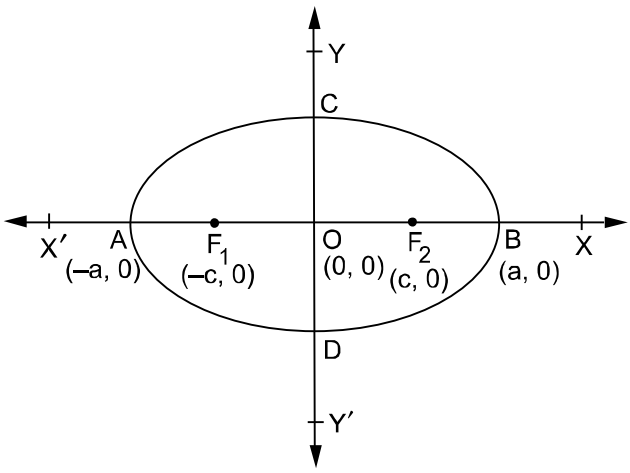
(i) The standard form of equation of a horizontal ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ 0 < b < a.$$

(ii) Its centre is $O(0, 0)$.

(iii) Its vertices are $A(-a, 0)$ and $B(a, 0)$.

(iv) Its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c^2 = (a^2 - b^2)$.



(v) Length of the major axis, $AB = 2a$ and length of the minor axis, $CD = 2b$.

(vi) Equation of the major axis is $y = 0$ and that of the minor axis is $x = 0$.

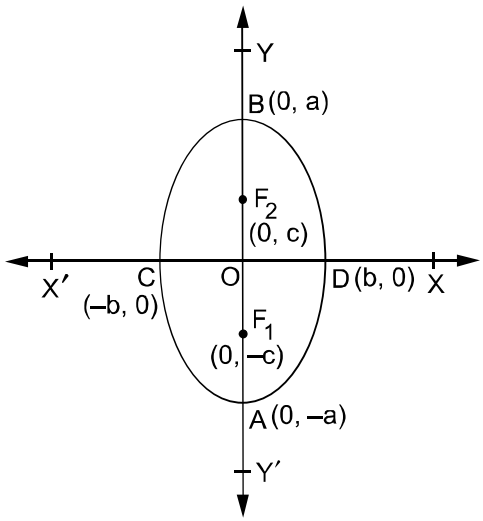
(vii) Length of the latus rectum $= \dfrac{2b^2}{a}$.

(viii) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}$.

VERTICAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the smaller denominator, then its major axis lies along the y-axis.

Such an ellipse is called a *vertical ellipse.*



RESULTS ON VERTICAL ELLIPSE

(i) The standard form of the equation of a vertical ellipse is $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $0 < b < a$.

(ii) Its centre is $O(0, 0)$.

(iii) Its vertices are $A(0, -a)$ and $B(0, a)$.

(iv) Its foci are $F_1(0, -c)$ and $F_2(0, c)$, where $a^2 - b^2 = c^2$,

i.e., $F_1(0, -ae)$ and $F_2(0, ae)$.

(v) Length of the major axis $= 2a$ and length of the minor axis $= 2b$.

(vi) Equation of the major axis is $x = 0$ and that of the minor axis is $y = 0$.

(vii) Length of latus rectum $= \dfrac{2b^2}{a}$.

(viii) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}$.

SUMMARY		
Properties	**Horizontal Ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **$0 < b < a$ and $c^2 = (a^2 - b^2)$**	**Vertical Ellipse** $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ **$0 < b < a$ and $c^2 = (a^2 - b^2)$**
(i) Centre	$(0, 0)$	$(0, 0)$
(ii) Vertices	$A(-a, 0)$, $B(a, 0)$	$A(0, -a)$ and $B(0, a)$
(iii) Foci	$F_1(-c, 0)$ and $F_2(c, 0)$ or $(-ae, 0)$ and $(ae, 0)$	$F_1(0, -c)$ and $F_2(0, c)$ or $(0, -ae)$ and $(0, ae)$
(iv) Length of the major axis	$2a$	$2a$
(v) Length of the minor axis	$2b$	$2b$
(vi) Equation of the major axis	$y = 0$	$x = 0$
(vii) Equation of the minor axis	$x = 0$	$y = 0$
(viii) Length of the latus rectum	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$
(ix) Eccentricity	$e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}$	$e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}$

SOLVED EXAMPLES

EXAMPLE 1 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci, the eccentricity and length of the latus rectum of the ellipse:*

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

SOLUTION Given equation is $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

This is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse.*

Now, $(a^2 = 16$ and $b^2 = 9 \Rightarrow (a = 4$ and $b = 3)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$.

Thus, $a = 4$, $b = 3$ and $c = \sqrt{7}$.

(i) Length of the major axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the minor axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-\sqrt{7}, 0)$ and $F_2(\sqrt{7}, 0)$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{7}}{4}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 9)}{3}$ units $= \dfrac{9}{2}$ units.

EXAMPLE 2 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$4x^2 + 9y^2 = 144.$$

SOLUTION The given equation may be written as

$$\frac{x^2}{36} + \frac{y^2}{16} = 1.$$

This is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse.*

Now, $(a^2 = 36$ and $b^2 = 16) \Rightarrow (a = 6$ and $b = 4)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}$.

Thus, $a = 6$, $b = 4$ and $c = 2\sqrt{5}$.

(i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 4)$ units $= 8$ units.

(ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-2\sqrt{5}, 0)$ and $F_2(2\sqrt{5}, 0)$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{2\sqrt{5}}{6} = \dfrac{\sqrt{5}}{3}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 16)}{6}$ units $= \dfrac{16}{3}$ units.

EXAMPLE 3 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$\frac{x^2}{4} + \frac{y^2}{36} = 1.$$

SOLUTION Given equation is $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$.

This is of the form $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse.*

Now, $(b^2 = 4$ and $a^2 = 36) \Rightarrow (b = 2$ and $a = 6)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = \sqrt{32} = 4\sqrt{2}.$

Thus, $a = 6$, $b = 2$ and $c = 4\sqrt{2}$.

(i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 2)$ units $= 4$ units.

(ii) Coordinates of its vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -6)$ and $B(0, 6)$.

(iii) Coordinates of its foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{4\sqrt{2}}{6} = \dfrac{2\sqrt{2}}{3}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 4)}{6}$ units $= \dfrac{4}{3}$ units.

EXAMPLE 4 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$4x^2 + y^2 = 100.$$

SOLUTION The given equation of the ellipse may be written as

$$\frac{x^2}{25} + \frac{y^2}{100} = 1.$$

This is of the form $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse.*

Now, $(b^2 = 25$ and $a^2 = 100) \Rightarrow (b = 5$ and $a = 10)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}.$

Thus, $a = 10$, $b = 5$ and $c = 5\sqrt{3}$.

(i) Length of the major axis $= 2a = (2 \times 10)$ units $= 20$ units.
Length of the minor axis $= 2b = (2 \times 5)$ units $= 10$ units.

(ii) Coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -10)$ and $B(0, 10)$.

(iii) Coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -5\sqrt{3})$ and $F_2(0, 5\sqrt{3})$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{5\sqrt{3}}{10} = \dfrac{\sqrt{3}}{2}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 25)}{10}$ units $= 5$ units.

EXAMPLE 5 *Find the equation of an ellipse whose vertices are at* $(\pm 5, 0)$ *and foci at* $(\pm 4, 0)$.

SOLUTION Since the vertices of the given ellipse are on the x-axis, so it is a *horizontal ellipse.*

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0)$ and therefore, $a = 5$.

Its foci are $(\pm c, 0)$ and therefore, $c = 4$.

$\therefore \quad c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (25 - 16) = 9.$

Thus, $a^2 = 5^2 = 25$ and $b^2 = 9$.

Hence, the required equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

EXAMPLE 6 *Find the equation of an ellipse whose foci are* $(\pm 4, 0)$ *and the eccentricity is* $\frac{1}{3}$.

SOLUTION Since the foci of the given ellipse are on the x-axis, so it is a *horizontal ellipse.*

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } a^2 > b^2.$$

Let its foci be $(\pm c, 0)$. Then, $c = 4$.

Also, $e = \frac{c}{a} \Leftrightarrow a = \frac{c}{e} = \frac{4}{(1/3)} = 12.$

Now, $c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (144 - 16) = 128.$

$\therefore \quad a^2 = (12)^2 = 144$ and $b^2 = 128$.

Hence, the required equation is $\frac{x^2}{144} + \frac{y^2}{128} = 1.$

EXAMPLE 7 *Find the equation of an ellipse whose major axis lies on the x-axis and which passes through the points* (4, 3) *and* (6, 2).

SOLUTION Since the major axis of the ellipse lies on the x-axis, so it is a *horizontal ellipse.*

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a^2 > b^2). \qquad \ldots \text{(i)}$$

Since (4, 3) lies on (i), we have $\frac{16}{a^2} + \frac{9}{b^2} = 1.$ $\ldots$ (ii)

Also, since (6, 2) lies on (i), we have $\frac{36}{a^2} + \frac{4}{b^2} = 1.$ $\ldots$ (iii)

Putting $\frac{1}{a^2} = u$ and $\frac{1}{b^2} = v$ in (ii) and (iii), we get

$$16u + 9v = 1 \qquad \ldots \text{(iv)}$$
$$36u + 4v = 1 \qquad \ldots \text{(v)}$$

On multiplying (iv) by 9 and (v) by 4, and subtracting, we get

$$65v = 5 \Leftrightarrow v = \frac{1}{13} \Leftrightarrow \frac{1}{b^2} = \frac{1}{13} \Leftrightarrow b^2 = 13.$$

Putting $v = \frac{1}{13}$ in (iv), we get

$$16u = \left(1 - \frac{9}{13}\right) \Leftrightarrow 16u = \frac{4}{13} \Leftrightarrow u = \left(\frac{4}{13} \times \frac{1}{16}\right) = \frac{1}{52}$$

$$\Leftrightarrow \frac{1}{a^2} = \frac{1}{52} \Leftrightarrow a^2 = 52.$$

Thus, $a^2 = 52$ and $b^2 = 13$.

Hence, the required equation is $\frac{x^2}{52} + \frac{y^2}{13} = 1$.

EXAMPLE 8 *Find the equation of the ellipse, the ends of whose major axis are* $(\pm 3, 0)$ *and the ends of whose minor axis are* $(0, \pm 2)$.

SOLUTION Clearly, the major axis of the given ellipse lies on the x-axis. So, it is a *horizontal ellipse.*

Let the required equation be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0)$ and, therefore, $a = 3$

[$\because$ ends of the major axis are the vertices].

Ends of the minor axis are $C(0, -2)$ and $D(0, 2)$.

$\therefore$ $CD = 4$, i.e., length of the minor axis = 4 units.

$\therefore$ $2b = 4 \Leftrightarrow b = 2$.

Now, $(a = 3$ and $b = 2) \Rightarrow (a^2 = 9$ and $b^2 = 4)$.

Hence, the required equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

EXAMPLE 9 *Find the equation of the ellipse whose centre lies at the origin, major axis lies on the x-axis, the eccentricity is* $\frac{2}{3}$ *and the length of the latus rectum is* 5 *units.*

SOLUTION Since the major axis of the ellipse lies on the x-axis, it is a horizontal ellipse.

Let the required equation be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Length of its latus rectum $= \frac{2b^2}{a}$.

$\therefore$ $\frac{2b^2}{a} = 5 \Leftrightarrow \frac{2a^2(1-e^2)}{a} = 5$ [$\because$ $b^2 = a^2(1-e^2)$]

$\Leftrightarrow$ $a = \frac{5}{2(1-e^2)} = \frac{5}{2\left(1-\frac{4}{9}\right)} = \left(\frac{5}{2} \times \frac{9}{5}\right) = \frac{9}{2}$.

Also, $b^2 = a^2(1-e^2) = \frac{81}{4} \times \left(1 - \frac{4}{9}\right) = \left(\frac{81}{4} \times \frac{5}{9}\right) = \frac{45}{4}$.

$\therefore$ $a^2 = \left(\frac{9}{2}\right)^2 = \frac{81}{4}$ and $b^2 = \frac{45}{4}$.

Hence, the required equation is

$$\frac{x^2}{(81/4)} + \frac{y^2}{(45/4)} = 1 \Leftrightarrow \frac{4x^2}{81} + \frac{4y^2}{45} = 1 \Leftrightarrow 20x^2 + 36y^2 = 405.$$

EXAMPLE 10 *Find the equation of an ellipse whose vertices are* $(0, \pm 13)$ *and the foci are* $(0, \pm 5)$.

SOLUTION Since the vertices of the ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a)$ and therefore, $a = 13$.

Let its foci be $(0, \pm c)$. Then, $c = 5$.

$\therefore \quad b^2 = (a^2 - c^2) = (169 - 25) = 144.$

Thus, $b^2 = 144$ and $a^2 = (13)^2 = 169$.

Hence, the required equation is $\frac{x^2}{144} + \frac{y^2}{169} = 1.$

EXAMPLE 11 *Find the equation of the ellipse whose foci are* $(0, \pm 6)$ *and the length of whose minor axis is* 16.

SOLUTION Since the foci of the given ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Then, its foci are $(0, \pm c)$ and therefore, $c = 6$.

Also, $b =$ length of the semi-minor axis $= \left(\frac{1}{2} \times 16\right) = 8.$

$\therefore \quad a^2 = (c^2 + b^2) = (36 + 64) = 100.$

Thus, $b^2 = 64$ and $a^2 = 100$.

Hence, the required equation is $\frac{x^2}{64} + \frac{y^2}{100} = 1.$

EXAMPLE 12 *Find the equation of the ellipse whose foci are* $(0, \pm 5)$ *and the length of whose major axis is* 20.

SOLUTION Since the foci of the ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Its foci are $(0, \pm c)$ and therefore, $c = 5$.

Also, $a =$ length of the semi-major axis $= \left(\frac{1}{2} \times 20\right) = 10.$

Now, $c^2 = (a^2 - b^2) \iff b^2 = (a^2 - c^2) = (100 - 25) = 75.$

Thus, $a^2 = (10)^2 = 100$ and $b^2 = 75$.

Hence, the required equation is $\frac{x^2}{75} + \frac{y^2}{100} = 1.$

EXAMPLE 13 *Find the equation of the ellipse for which* $e = \frac{4}{5}$ *and whose vertices are* $(0, \pm 10)$.

SOLUTION Since the vertices of the ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a)$ and therefore, $a = 10$.

Let $c^2 = (a^2 - b^2)$.

Then, $e = \frac{c}{a} \Rightarrow c = ae = \left(10 \times \frac{4}{5}\right) = 8$.

Now, $c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 64) = 36$.

$\therefore \quad a^2 = (10)^2 = 100$ and $b^2 = 36$.

Hence, the required equation is $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

EXAMPLE 14 *Find the equation of the ellipse with centre at the origin, major-axis on the y-axis and passing through the points* (3, 2) *and* (1, 6).

SOLUTION Since the major axis of the ellipse lies on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (where $a^2 > b^2$). … (i)

Since (3, 2) lies on (i), we have $\frac{9}{b^2} + \frac{4}{a^2} = 1$. … (ii)

Also, since (1, 6) lies on (i), we have $\frac{1}{b^2} + \frac{36}{a^2} = 1$. … (iii)

Putting $\frac{1}{b^2} = u$ and $\frac{1}{a^2} = v$, these equations become:

$9u + 4v = 1$ … (iv)

and $u + 36v = 1$ … (v)

On multiplying (v) by 9 and subtracting (iv) from it, we get

$$320v = 8 \Leftrightarrow v = \frac{8}{320} = \frac{1}{40} \Leftrightarrow \frac{1}{a^2} = \frac{1}{40} \Leftrightarrow a^2 = 40.$$

Putting $v = \frac{1}{40}$ in (v), we get

$$u + \left(36 \times \frac{1}{40}\right) = 1 \Leftrightarrow u = \left(1 - \frac{9}{10}\right) = \frac{1}{10} \Leftrightarrow \frac{1}{b^2} = \frac{1}{10} \Leftrightarrow b^2 = 10.$$

Thus, $b^2 = 10$ and $a^2 = 40$.

Hence, the required equation is $\frac{x^2}{10} + \frac{y^2}{40} = 1$.

EXERCISE 23

Find the (i) lengths of major and minor axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity, and (v) length of the latus rectum of each of the following ellipses.

1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **2.** $\frac{x^2}{49} + \frac{y^2}{36} = 1$ **3.** $16x^2 + 25y^2 = 400$

4. $x^2 + 4y^2 = 100$ **5.** $9x^2 + 16y^2 = 144$ **6.** $4x^2 + 9y^2 = 1$

7. $\frac{x^2}{4} + \frac{y^2}{25} = 1$ **8.** $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **9.** $3x^2 + 2y^2 = 18$

10. $9x^2 + y^2 = 36$ **11.** $16x^2 + y^2 = 16$ **12.** $25x^2 + 4y^2 = 100$

13. Find the equation of the ellipse whose vertices are at $(\pm 6, 0)$ and foci at $(\pm 4, 0)$.

14. Find the equation of the ellipse whose vertices are at $(0, \pm 4)$ and foci at $(0, \pm\sqrt{7})$.

15. Find the equation of the ellipse the ends of whose major and minor axes are $(\pm 4, 0)$ and $(0, \pm 3)$ respectively.

16. The length of the major axis of an ellipse is 20 units and its foci are $(\pm 5\sqrt{3}, 0)$. Find the equation of the ellipse.

17. Find the equation of the ellipse whose foci are $(\pm 2, 0)$ and the eccentricity is $\frac{1}{2}$.

18. Find the equation of the ellipse whose foci are at $(\pm 1, 0)$ and $e = \frac{1}{2}$.

19. Find the equation of the ellipse whose foci are at $(0, \pm 4)$ and $e = \frac{4}{5}$.

20. Find the equation of the ellipse with centre at the origin, major axis on the x-axis and passing through the points $(4, 3)$ and $(-1, 4)$.

21. Find the equation of the ellipse with eccentricity $\frac{3}{4}$, foci on the y-axis, centre at the origin and passing through the point $(6, 4)$.

22. Find the equation of the ellipse which passes through the point $(4, 1)$ and having its foci at $(\pm 3, 0)$.

23. Find the equation of an ellipse, the lengths of whose major and minor axes are 10 and 8 units respectively.

24. Find the equation of an ellipse whose eccentricity is $\frac{2}{3}$, the latus rectum is 5 and the centre is at the origin.

25. Find the eccentricity of an ellipse whose latus rectum is one half of its minor axis.

26. Find the eccentricity of an ellipse whose latus rectum is one half of its major axis.

ANSWERS (EXERCISE 23)

1. (i) 10 units, 6 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-4, 0)$ and $F_2(4, 0)$

(iv) $e = \frac{4}{5}$ (v) 3.6 units

2. (i) 14 units, 12 units (ii) $A(-7, 0)$ and $B(7, 0)$

(iii) $F_1(-\sqrt{13}, 0)$ and $F_2(\sqrt{13}, 0)$ (iv) $e = \frac{\sqrt{13}}{7}$ (v) $\frac{72}{7}$ units

3. (i) 10 units, 8 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-3, 0)$ and $F_2(3, 0)$

(iv) $e = \frac{3}{5}$ (v) $\frac{32}{5}$ units

4. (i) 20 units, 10 units (ii) $A(-10, 0)$ and $B(10, 0)$

(iii) $F_1(-5\sqrt{3}, 0)$ and $F_2(5\sqrt{3}, 0)$ (iv) $e = \frac{\sqrt{3}}{2}$ (v) 5 units

5. (i) 8 units, 6 units (ii) $A(-4, 0)$ and $B(4, 0)$ (iii) $F_1(-\sqrt{7}, 0)$, $F_2(\sqrt{7}, 0)$

(iv) $e = \frac{\sqrt{7}}{4}$ (v) $\frac{9}{2}$ units

6. (i) 1 unit, $\frac{2}{3}$ unit (ii) $A\left(-\frac{1}{2}, 0\right)$ and $B\left(\frac{1}{2}, 0\right)$

(iii) $F_1\left(\frac{-\sqrt{5}}{6}, 0\right)$ and $F_2\left(\frac{\sqrt{5}}{6}, 0\right)$ (iv) $e = \frac{\sqrt{5}}{3}$ (v) $\frac{4}{9}$ unit

7. (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$

(iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $\frac{8}{5}$ units

8. (i) 8 units, 6 units (ii) $A(0, -4)$ and $B(0, 4)$

(iii) $F_1(0, -\sqrt{7})$ and $F_2(0, \sqrt{7})$ (iv) $e = \frac{\sqrt{7}}{4}$ (v) $4\frac{1}{2}$ units

9. (i) 6 units, $2\sqrt{6}$ units (ii) $A(0, -3)$ and $B(0, 3)$

(iii) $F_1(0, -\sqrt{3})$ and $F_2(0, \sqrt{3})$ (iv) $e = \frac{1}{\sqrt{3}}$ (v) 4 units

10. (i) 12 units, 4 units (ii) $A(0, -6)$ and $B(0, 6)$

(iii) $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$ (iv) $e = \frac{2\sqrt{2}}{3}$ (v) $1\frac{1}{3}$ units

11. (i) 8 units, 2 units (ii) $A(0, -4)$ and $B(0, 4)$

(iii) $F_1(0, -\sqrt{15})$ and $F_2(0, \sqrt{15})$ (iv) $e = \frac{\sqrt{15}}{4}$ (v) $\frac{1}{2}$ unit

12. (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$

(iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $1\frac{3}{5}$ units

13. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ **14.** $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **15.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$

16. $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **17.** $\frac{x^2}{16} + \frac{y^2}{12} = 1$ **18.** $\frac{x^2}{4} + \frac{y^2}{3} = 1$

19. $\frac{x^2}{9} + \frac{y^2}{25} = 1$ **20.** $7x^2 + 15y^2 = 247$ **21.** $16x^2 + 7y^2 = 688$

22. $\frac{x^2}{18} + \frac{y^2}{9} = 1$ **23.** $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **24.** $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

25. $e = \frac{\sqrt{3}}{2}$ **26.** $e = \frac{1}{\sqrt{2}}$

HINTS TO SOME SELECTED QUESTIONS

6. Given equation is $\frac{x^2}{(1/4)} + \frac{y^2}{(1/9)} = 1$.

$\therefore \quad a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$ and $c^2 = (a^2 - b^2) = \left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36}$.

$\therefore \quad a = \frac{1}{2}, b = \frac{1}{3}$ and $c = \frac{\sqrt{5}}{6}$.

15. The ends of the major axis are $(\pm a, 0)$ and the ends of the minor axis are $(0, \pm b)$.

$\therefore \quad a = 4$ and $b = 3$.

16. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Length of the major axis = 20 $\Leftrightarrow$ $2a = 20$ $\Leftrightarrow$ $a = 10$.

Foci are $(\pm c, 0)$ $\Rightarrow$ $c = 5\sqrt{3}$.

Now, $c^2 = (a^2 - b^2)$ $\Leftrightarrow$ $b^2 = (a^2 - c^2) = (100 - 75) = 25$.

$\therefore \quad$ the required equation is $\frac{x^2}{100} + \frac{y^2}{25} = 1$.

19. Clearly, the major axis lies along the y-axis.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Here, $c = 4$ and $e = \frac{4}{5}$. So, $a = \frac{c}{e} = 5$.

$\therefore \quad b^2 = (a^2 - c^2) = (25 - 16) = 9.$

21. Since the centre of the ellipse lies at the origin and its foci lie on the y-axis, it is a vertical ellipse.

Let its equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$ … (i)

Let $c^2 = a^2 - b^2$.

Now, $e = \frac{3}{4} \Leftrightarrow \frac{c}{a} = \frac{3}{4} \Leftrightarrow c = \frac{3}{4}a.$

$\therefore \quad b^2 = (a^2 - c^2) = \left(a^2 - \frac{9}{16}a^2\right) = \frac{7a^2}{16}.$

So, the equation of the ellipse is

$$\frac{x^2}{\frac{7a^2}{16}} + \frac{y^2}{a^2} = 1 \Leftrightarrow 16x^2 + 7y^2 = 7a^2. \quad \text{... (ii)}$$

Since it passes through (6, 4), putting $x = 6$ and $y = 4$ in (ii), we get

$7a^2 = 688.$

Hence, the required equation is $16x^2 + 7y^2 = 688.$

23. $2a = 10$ and $2b = 8$.

24. $\frac{2b^2}{a} = 5 \Rightarrow \frac{2a^2(1 - e^2)}{a} = 5 \Rightarrow a = \frac{9}{2}.$

Also, $b^2 = \frac{5a}{2} = \left(\frac{5}{2} \times \frac{9}{2}\right) = \frac{45}{4}.$

25. $\left\{\frac{2b^2}{a} = \frac{1}{2} \times 2b \Rightarrow b = \frac{a}{2}\right\}.$

So, $b^2 = a^2(1 - e^2) \Rightarrow \frac{a^2}{4} = a^2(1 - e^2).$

26. $\frac{2b^2}{a} = \frac{1}{2} \times 2a \Rightarrow a^2 = 2b^2 \Rightarrow a = \sqrt{2}\, b.$ So, $e = \frac{b}{a} = \frac{1}{\sqrt{2}}.$

24
Hyperbola

HYPERBOLA *It is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.*

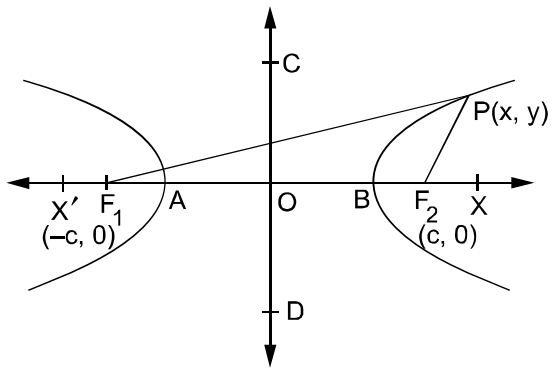
The two fixed points are called the *foci* of the hyperbola.

SOME MORE TERMS RELATED TO A HYPERBOLA

(I) CENTRE OF THE HYPERBOLA

The midpoint of the line segment joining the foci is called the centre of the hyperbola.

In the given figure, F_1 and F_2 are the foci of the hyperbola and O is its centre, where $OF_1 = OF_2$.



(II) AXES OF THE HYPERBOLA

TRANSVERSE AXIS: *The line through the foci of the hyperbola is called its transverse axis.*

In the given figure, $X'OX$ is the transverse axis of the hyperbola.

CONJUGATE AXIS: *The line through the centre and perpendicular to the transverse axis of the hyperbola is called its conjugate axis.*

Thus, COD is the conjugate axis of the hyperbola in the given figure.

(III) VERTICES OF THE HYPERBOLA

The points at which the hyperbola intersects the transverse axis are called its vertices.

In the given figure, A and B are the vertices of the hyperbola.

(IV) LENGTH OF TRANSVERSE AXIS

The distance between the two vertices of a hyperbola is called the length of its transverse axis.

In the given hyperbola, length of the transverse axis = AB.

AN IMPORTANT NOTE In a hyperbola, we take:

Length of the transverse axis $= AB = 2a$.

Distance between the foci $= F_1F_2 = 2c$, where $c > a$.

Length of the conjugate axis $= 2b$, where $b^2 = (c^2 - a^2)$.

(V) ECCENTRICITY OF A HYPERBOLA

The ratio $\frac{c}{a}$ *is always constant, called the eccentricity of the hyperbola and it is denoted by e.*

Here, $c > a \Leftrightarrow \frac{c}{a} > 1 \Leftrightarrow e > 1$.

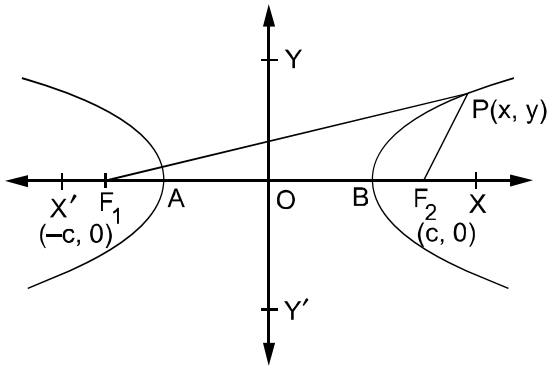
STANDARD EQUATION OF A HYPERBOLA

THEOREM *Prove that the standard equation of a hyperbola is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let us consider a hyperbola with centre at $O(0, 0)$ and its foci at $F_1(-c, 0)$ and $F_2(c, 0)$.



Let $P(x, y)$ be an arbitrary point on the hyperbola such that $PF_1 - PF_2 = 2a$. Then,

$$PF_1 - PF_2 = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 + 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow (x+c)^2 - (x-c)^2 - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4cx - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \left(\frac{cx}{a} - a\right) = \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \left(\frac{cx}{a} - a\right)^2 = (x-c)^2 + y^2$$

$$\Rightarrow \quad \frac{c^2x^2}{a^2} + a^2 = x^2 + c^2 + y^2$$

$$\Rightarrow \quad x^2\left(\frac{c^2}{a^2} - 1\right) - y^2 = (c^2 - a^2)$$

$$\Rightarrow \quad x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{(c^2 - a^2)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } (c^2 - a^2) = b^2.$$

Thus, every point on the given hyperbola satisfies the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Conversely, let the equation of the given curve be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and let $P(x, y)$ be an arbitrary point on it. Then,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \;\Rightarrow\; y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$

$$\Rightarrow \quad y^2 = \frac{b^2(x^2 - a^2)}{a^2}. \qquad \ldots \text{(i)}$$

Let $c^2 = (a^2 + b^2)$. $\qquad \ldots$ (ii)

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be two fixed points on the x-axis. Then,

$$PF_1 = \sqrt{(x + c)^2 + y^2}$$

$$= \sqrt{(x + c^2) + \frac{b^2(x^2 - a^2)}{a^2}} \qquad \text{[using (i)]}$$

$$= \sqrt{(x + c)^2 + \frac{(c^2 - a^2)(x^2 - a^2)}{a^2}} \qquad \text{[using (ii)}$$

$$= \sqrt{(x^2 + c^2 + 2cx) + \frac{(c^2x^2 - c^2a^2 - a^2x^2 + a^4)}{a^2}}$$

$$= \sqrt{(x^2 + c^2 + 2cx) + \frac{c^2x^2}{a^2} - c^2 - x^2 + a^2}$$

$$= \sqrt{\frac{c^2x^2}{a^2} + a^2 + 2cx} = \sqrt{\left(\frac{cx}{a} + a\right)^2} = \left(\frac{cx}{a} + a\right).$$

Similarly, $PF_2 = \left(\frac{cx}{a} - a\right)\left[x \geq a \text{ and } c \geq a \;\Rightarrow\; \frac{x}{a} \geq 1 \text{ and } c \geq a \;\Rightarrow\; \frac{cx}{a} \geq a\right]$

$$\therefore \quad PF_1 - PF_2 = \left(\frac{cx}{a} + a\right) - \left(\frac{cx}{a} - a\right) = 2a.$$

This shows that P lies on the hyperbola.

Thus, every point that satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, lies on the hyperbola.

Hence, the equation of a hyperbola with origin (0, 0) and transverse axis along the x-axis is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

REMARK A hyperbola in which $a = b$ is called an equilateral hyperbola.

LATUS RECTUM OF A HYPERBOLA

The latus rectum of a hyperbola is a line segment perpendicular to the transverse axis, through any of the foci with its end points lying on the hyperbola.

TO FIND THE LENGTH OF LATUS RECTUM OF THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

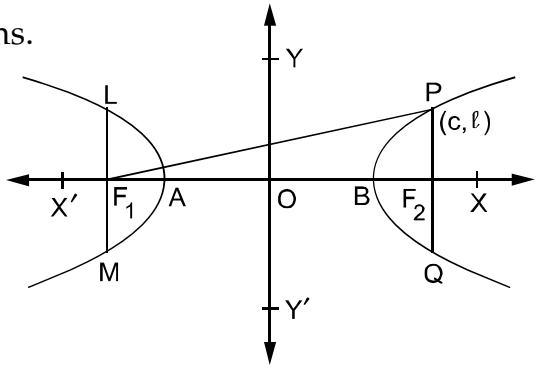
Let $X'OX$ and YOY' be the coordinate axes.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the given hyperbola, where $c^2 = (a^2 + b^2)$.

Let LF_1M and PF_2Q be the latus rectums.

Let $PF_2 = l$.

Then, the coordinates of P are $P(c, l)$.



Since $P(c, l)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$\frac{c^2}{a^2} - \frac{l^2}{b^2} = 1 \Rightarrow \frac{l^2}{b^2} = \left(\frac{c^2}{a^2} - 1\right)$$

$$\Rightarrow l^2 = b^2\left(\frac{c^2}{a^2} - 1\right) = b^2 \cdot \left\{\frac{(a^2 + b^2)}{a^2} - 1\right\} \quad [\because\ c^2 = (a^2 + b^2)]$$

$$\Rightarrow l^2 = \frac{b^4}{a^2}$$

$$\Rightarrow l = \frac{b^2}{a}.$$

$$\therefore\ PF_2 = l = \frac{b^2}{a} \Leftrightarrow PF_2Q = 2l = \frac{2b^2}{a}.$$

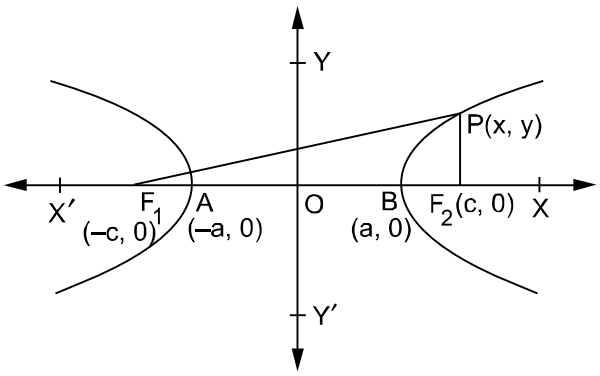
Similarly, $LF_1M = 2l = \frac{2b^2}{a}$.

Hence, the length of the latus rectum is $\frac{2b^2}{a}$.

SUMMARY OF THE RESULTS ON HORIZONTAL HYPERBOLA

(i) The standard equation of a horizontal hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(ii) Its *centre* is $O(0, 0)$.



(iii) $X'OX$ is the *transverse axis* and YOY' is the *conjugate axis.*

(iv) Its *foci* are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-ae, 0)$ and $F_2(ae, 0)$.

(v) Its *vertices* are $A(-a, 0)$ and $B(a, 0)$.

(vi) Its *eccentricity* is, $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a}$.

(vii) Length of the transverse axis = $2a$ and its equation is $y = 0$.

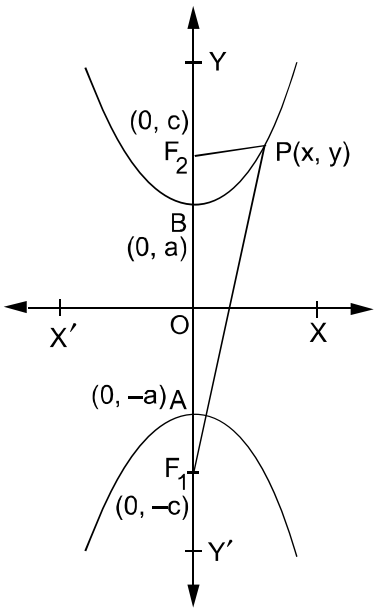
(viii) Length of the conjugate axis = $2b$ and its equation is $x = 0$.

(ix) Length of its *latus rectum* = $\dfrac{2b^2}{a}$.

We have similar results for *vertical hyperbola,* given below.

SUMMARY OF THE RESULTS ON VERTICAL HYPERBOLA

(i) The standard equation of a vertical hyperbola is $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.



(ii) Its *centre* is $O(0, 0)$.

(iii) YOY' is the *transverse axis* and $X'OX$ is the *conjugate axis.*

(iv) Its *foci* are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -ae)$ and $F_2(0, ae)$.

(v) Its *vertices* are $A(0, -a)$ and $B(0, a)$.

(vi) Its *eccentricity* is, $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a}$.

(vii) Length of the transverse axis = $2a$ and its equation is $x = 0$.

(viii) Length of the conjugate axis = $2b$ and its equation is $y = 0$.

(ix) Length of its *latus rectum* = $\dfrac{2b^2}{a}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$\frac{x^2}{36} - \frac{y^2}{64} = 1.$$

SOLUTION The equation of the given hyperbola is $\frac{x^2}{36} - \frac{y^2}{64} = 1.$

Comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 36$ and $b^2 = 64.$

$\therefore$ $a = 6,\ b = 8$ and $c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$

(i) Length of the transverse axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the conjugate axis $= 2b = (2 \times 8)$ units $= 16$ units.

(ii) The coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iii) The coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-10, 0)$ and $F_2(10, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 64}{6}\right)$ units $= \frac{64}{3}$ units.

EXAMPLE 2 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$9x^2 - 16y^2 = 144.$$

SOLUTION $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1.$

Thus, the equation of the given hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

Comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 16$ and $b^2 = 9.$

$\therefore$ $a = 4,\ b = 3$ and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$

(i) Length of the transverse axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the conjugate axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) The coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iii) The coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-5, 0)$ and $F_2(5, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{5}{4}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 9}{4}\right)$ units $= \frac{9}{2}$ units.

EXAMPLE 3 *Find the equation of the hyperbola whose vertices are* $(\pm 2, 0)$ *and the foci are* $(\pm 3, 0)$.

SOLUTION Since the vertices of the given hyperbola are of the form $(\pm a, 0)$, it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, its vertices are $(\pm a, 0)$.

But, it is given that the vertices are $(\pm 2, 0)$.

$\therefore \quad a = 2$.

Let its foci be $(\pm c, 0)$.

But, it is given that the foci are $(\pm 3, 0)$.

$\therefore \quad c = 3$.

Now, $b^2 = (c^2 - a^2) = (3^2 - 2^2) = (9 - 4) = 5$.

Thus, $a^2 = 2^2 = 4$ and $b^2 = 5$.

Hence, the required equation is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

EXAMPLE 4 *Find the equation of the hyperbola whose foci are* $(\pm 5, 0)$ *and the transverse axis is of length* 8.

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c, 0)$, it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Length of its transverse axis $= 2a$.

$\therefore \quad 2a = 8 \Leftrightarrow a = 4 \Leftrightarrow a^2 = 16$.

Let its foci be $(\pm c, 0)$.

Then, $c = 5$ $\quad [\because$ foci are $(\pm 5, 0)]$.

$\therefore \quad b^2 = (c^2 - a^2) = (5^2 - 4^2) = (25 - 16) = 9$.

Thus, $a^2 = 16$ and $b^2 = 9$.

Hence, the required equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

EXAMPLE 5 *Find the equation of the hyperbola whose foci are* $(\pm 4, 0)$ *and the length of the latus rectum is* 12 *units.*

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c, 0)$, it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let its foci be $(\pm c, 0)$.

Then, $c = 4$ $\quad [\because$ foci are $(\pm 4, 0)]$.

Now, $c = 4 \Leftrightarrow c^2 = 16 \Leftrightarrow a^2 + b^2 = 16.$ … (i)

Length of the latus rectum $= 12 \Leftrightarrow \frac{2b^2}{a} = 12 \Leftrightarrow b^2 = 6a.$ … (ii)

From (i) and (ii), we get

$$a^2 + 6a = 16 \Leftrightarrow a^2 + 6a - 16 = 0$$
$$\Leftrightarrow (a + 8)(a - 2) = 0$$
$$\Leftrightarrow a = 2 \quad [\because a \text{ cannot be negative}].$$

Putting $a = 2$ in (ii), we get $b^2 = (6 \times 2) = 12.$

Thus, $a^2 = 2^2 = 4$ and $b^2 = 12.$

Hence, the required equation is $\frac{x^2}{4} - \frac{y^2}{12} = 1.$

EXAMPLE 6 *Find the equation of the hyperbola whose vertices are* $(\pm 7, 0)$ *and the eccentricity is* $\frac{4}{3}$.

SOLUTION Since the vertices of the given hyperbola are of the form $(\pm a, 0)$, it is a horizontal hyperbola.

Let the required equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Then, its vertices are $(\pm a, 0)$.

But, the vertices are $(\pm 7, 0)$.

$\therefore \quad a = 7 \Leftrightarrow a^2 = 49.$

Also, $e = \frac{c}{a} \Leftrightarrow c = ae = \left(7 \times \frac{4}{3}\right) = \frac{28}{3}.$

Now, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = \left[\left(\frac{28}{3}\right)^2 - 49\right] = \frac{343}{9}.$

Thus, $a^2 = 49$ and $b^2 = \frac{343}{9}.$

$\therefore$ the required equation is

$$\frac{x^2}{49} - \frac{y^2}{(343/9)} = 1 \Leftrightarrow \frac{x^2}{49} - \frac{9y^2}{343} = 1 \Leftrightarrow 7x^2 - 9y^2 = 343.$$

EXAMPLE 7 *Find the equation of the hyperbola whose eccentricity is* $\frac{3}{2}$ *and whose foci are* $(\pm 2, 0)$.

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c, 0)$, it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Let the foci be $(\pm c, 0)$.

Then, $c = 2 \quad [\because$ foci are $(\pm 2, 0)]$.

$$\therefore \quad e = \frac{c}{a} \Leftrightarrow a = \frac{c}{e} = \frac{2}{(3/2)} = \frac{4}{3}.$$

And, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = \left(4 - \frac{16}{9}\right) = \frac{20}{9}.$

Thus, $a^2 = \frac{16}{9}$ and $b^2 = \frac{20}{9}.$

Hence, the required equation is

$$\frac{x^2}{\left(\frac{16}{9}\right)} = \frac{y^2}{\left(\frac{20}{9}\right)} = 1 \Leftrightarrow \frac{9x^2}{16} - \frac{9y^2}{20} = 1 \Leftrightarrow 45x^2 - 36y^2 = 80.$$

EXAMPLE 8 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$\frac{y^2}{4} - \frac{x^2}{9} = 1.$$

SOLUTION The given equation is $\frac{y^2}{4} - \frac{x^2}{9} = 1.$

Clearly, the given equation represents a vertical hyperbola.

Comparing the given equation with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we get

$$(a^2 = 4 \text{ and } b^2 = 9) \Rightarrow (a = 2 \text{ and } b = 3).$$

$$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13}.$$

(i) Length of the transverse axis $= 2a = (2 \times 2)$ units $= 4$ units.
Length of the conjugate axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) The coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -2)$ and $B(0, 2)$.

(iii) The coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -\sqrt{13})$ and $F_2(0, \sqrt{13})$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}.$

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 9}{2}\right)$ units $= 9$ units.

EXAMPLE 9 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$y^2 - 16x^2 = 16.$$

SOLUTION $y^2 - 16x^2 = 16 \Rightarrow \frac{y^2}{16} - \frac{x^2}{1} = 1.$

Clearly, the given equation represents a vertical hyperbola.

Comparing the given equation with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we get

$$(a^2 = 16 \text{ and } b^2 = 1) \Rightarrow (a = 4 \text{ and } b = 1).$$

$$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{16 + 1} = \sqrt{17}.$$

(i) Length of the transverse axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the conjugate axis $= 2b = (2 \times 1)$ units $= 2$ units.

(ii) The coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -4)$ and $B(0, 4)$.

(iii) The coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -\sqrt{17})$ and $F_2(0, \sqrt{17})$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{17}}{4}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \left(\dfrac{2 \times 1}{4}\right)$ unit $= \dfrac{1}{2}$ unit.

EXAMPLE 10 *Find the equation of the hyperbola whose vertices are* $(0, \pm 3)$ *and the foci are* $(0, \pm 5)$.

SOLUTION Since the vertices of the given hyperbola are of the form $(0, \pm a)$, it is a vertical hyperbola.

Let the required equation be $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

Then, its vertices are $(0, \pm a)$.

But, it is given that the vertices are $(0, \pm 3)$.

$\therefore \quad a = 3$.

Let its foci be $(0, \pm c)$.

But, it is given that the foci are $(0, \pm 5)$.

$\therefore \quad c = 5$.

Now, $b^2 = (c^2 - a^2) = (5^2 - 3^2) = (25 - 9) = 16$.

Thus, $a^2 = 3^2 = 9$ and $b^2 = 16$.

Hence, the required equation is $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$.

EXAMPLE 11 *Find the equation of the hyperbola whose eccentricity is* $\dfrac{5}{3}$ *and whose vertices are* $(0, \pm 6)$. *Also, find the coordinates of its foci.*

SOLUTION Since the vertices of the given hyperbola are of the form $(0, \pm a)$, it is a vertical hyperbola.

Let the required equation be $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

Then, its vertices are $(0, \pm a)$.

But, it is given that the vertices are $(0, \pm 6)$.

$\therefore \quad a = 6$. Also, $e = \dfrac{5}{3}$.

Now, $\dfrac{c}{a} = e \;\Rightarrow\; c = ae = \left(6 \times \dfrac{5}{3}\right) = 10$.

And, $c^2 = (a^2 + b^2) \;\Leftrightarrow\; b^2 = (c^2 - a^2) = \{(10)^2 - 6^2\} = (100 - 36) = 64$.

Thus, $a^2 = 6^2 = 36$ and $b^2 = 64$.

Hence, the required equation is $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

Also, the coordinates of its foci are $(0, \pm c)$, i.e., $(0, \pm 10)$.

EXAMPLE 12 *Find the equation of the hyperbola whose foci are* $(0, \pm 12)$ *and the length of whose latus rectum is* 36.

SOLUTION Since the coordinates of the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let the required equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Clearly, $c = 12$.

Length of the latus rectum $= 36 \Leftrightarrow \frac{2b^2}{a} = 36 \Leftrightarrow b^2 = 18a$.

Now, $c^2 = (a^2 + b^2) \Leftrightarrow a^2 = (c^2 - b^2) = (144 - 18a)$

$\Leftrightarrow a^2 + 18a - 144 = 0 \Leftrightarrow (a + 24)(a - 6) = 0$

$\Leftrightarrow a = 6$ [$\because$ a is non-negative].

Thus, $a^2 = 6^2 = 36$ and $b^2 = 18a = (18 \times 6) = 108$.

Hence, the required equation is $\frac{x^2}{36} - \frac{y^2}{108} = 1$.

EXAMPLE 13 *Find the equation of the hyperbola with centre at the origin, length of the transverse axis 6 and one focus at* $(0, 4)$.

SOLUTION Since the coordinates of the foci are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Clearly, $c = 4$ [$\because$ $(0, c) = (0, 4)$].

Length of the transverse axis $= 6 \Leftrightarrow 2a = 6 \Leftrightarrow a = 3$.

Also, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = (4^2 - 3^2) = (16 - 9) = 7$.

Thus, $a^2 = 3^2 = 9$ and $b^2 = 7$.

Hence, the required equation is $\frac{y^2}{9} - \frac{x^2}{7} = 1$.

EXAMPLE 14 *Find the equation of the hyperbola whose foci are at* $(0, \pm 6)$ *and the length of whose conjugate axis is* $2\sqrt{11}$.

SOLUTION Since the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Let its foci be $(0, \pm c)$.

But, the foci are at $(0, \pm 6)$.

$\therefore \quad c = 6.$

Length of the conjugate axis $= 2\sqrt{11}$

$2b = 2\sqrt{11} \Leftrightarrow b = \sqrt{11} \Leftrightarrow b^2 = 11.$

Also, $c^2 = (a^2 + b^2) \Leftrightarrow a^2 = (c^2 - b^2) = (36 - 11) = 25.$

Thus, $a^2 = 25$ and $b^2 = 11.$

Hence, the required equation is $\frac{y^2}{25} - \frac{x^2}{11} = 1.$

EXAMPLE 15 *Find the equation of the hyperbola whose foci are at* $(0, \pm\sqrt{10})$ *and which passes through the point* $(2, 3)$.

SOLUTION Since the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$ ... (i)

Let its foci be $(0, \pm c)$.

But, the foci are $(0, \pm\sqrt{10})$.

$\therefore \quad c = \sqrt{10} \Leftrightarrow c^2 = 10 \Leftrightarrow (a^2 + b^2) = 10$ ... (ii)

$[\because \ c^2 = (a^2 + b^2)].$

Since (i) passes through (2, 3), we have $\frac{9}{a^2} - \frac{4}{b^2} = 1.$

Now, $\frac{9}{a^2} - \frac{4}{b^2} = 1 \Leftrightarrow \frac{9}{a^2} - \frac{4}{(10 - a^2)} = 1$ [using (ii)]

$\Leftrightarrow 9(10 - a^2) - 4a^2 = a^2(10 - a^2)$

$\Leftrightarrow a^4 - 23a^2 + 90 = 0$

$\Leftrightarrow (a^2 - 18)(a^2 - 5) = 0$

$\Leftrightarrow a^2 = 5$

$[\because \ a^2 = 18 \Rightarrow b^2 = -8,$ which is not possible$].$

Thus, $a^2 = 5$ and $b^2 = 5.$

Hence, the required equation is $\frac{y^2}{5} - \frac{x^2}{5} = 1,$ i.e., $y^2 - x^2 = 5.$

EXERCISE 24

Find the (i) lengths of the axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity and (iv) length of the latus rectum of each of the following the hyperbola:

1. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

2. $\frac{x^2}{25} - \frac{y^2}{4} = 1$

3. $x^2 - y^2 = 1$

4. $3x^2 - 2y^2 = 6$

5. $25x^2 - 9y^2 = 225$ **6.** $24x^2 - 25y^2 = 600$ **7.** $\frac{y^2}{16} - \frac{x^2}{49} = 1$

8. $\frac{y^2}{9} - \frac{x^2}{27} = 1$ **9.** $3y^2 - x^2 = 108$ **10.** $5y^2 - 9x^2 = 36$

11. Find the equation of the hyperbola with vertices at $(\pm 6, 0)$ and foci at $(\pm 8, 0)$.

12. Find the equation of the hyperbola with vertices at $(0, \pm 5)$ and foci at $(0, \pm 8)$.

13. Find the equation of the hyperbola whose foci are $(\pm\sqrt{29}, 0)$ and the transverse axis is of the length 10.

14. Find the equation of the hyperbola whose foci are $(\pm 5, 0)$ and the conjugate axis is of the length 8. Also, find its eccentricity.

15. Find the equation of the hyperbola whose foci are $(\pm 3\sqrt{5}, 0)$ and the length of the latus rectum is 8 units.

16. Find the equation of the hyperbola whose vertices are $(\pm 2, 0)$ and the eccentricity is 2.

17. Find the equation of the hyperbola whose foci are $(\pm\sqrt{5}, 0)$ and the eccentricity is $\sqrt{\frac{5}{3}}$.

18. Find the equation of the hyperbola, the length of whose latus rectum is 4 and the eccentricity is 3.

19. Find the equation of the hyperbola with eccentricity $\sqrt{2}$ and the distance between whose foci is 16.

20. Find the equation of the hyperbola whose vertices are $(0, \pm 3)$ and the eccentricity is $\frac{4}{3}$. Also find the coordinates of its foci.

21. Find the equation of the hyperbola whose foci are $(0, \pm 13)$ and the length of whose conjugate axis is 24.

22. Find the equation of the hyperbola whose foci are $(0, \pm 10)$ and the length of whose latus rectum is 9 units.

23. Find the equation of the hyperbola having its foci at $(0, \pm\sqrt{14})$ and passing through the point $P(3, 4)$.

ANSWERS (EXERCISE 24)

1. (i) 6 units, 8 units (ii) $A(-3, 0)$, $B(3, 0)$ (iii) $F_1(-5, 0)$, $F_2(5, 0)$

(iv) $e = \frac{5}{3}$ (v) $10\frac{2}{3}$ units

2. (i) 10 units, 4 units (ii) $A(-5, 0)$, $B(5, 0)$ (iii) $F_1(-\sqrt{29}, 0)$, $F_2(\sqrt{29}, 0)$

(iv) $e = \frac{\sqrt{29}}{5}$ (v) $1\frac{3}{5}$ units

3. (i) 2 units, 2 units (ii) $A(-1, 0)$, $B(1, 0)$ (iii) $F_1(-\sqrt{2}, 0)$, $F_2(\sqrt{2}, 0)$

(iv) $e = \sqrt{2}$ (v) 2 units

4. (i) $2\sqrt{2}$ units, $2\sqrt{3}$ units (ii) $A(-\sqrt{2}, 0)$, $B(\sqrt{2}, 0)$ (iii) $F_1(-\sqrt{5}, 0)$, $F_2(\sqrt{5}, 0)$

(iv) $e = \sqrt{\frac{5}{2}}$ (v) $3\sqrt{2}$ units

5. (i) 6 units, 10 units (ii) $A(-3, 0)$, $B(3, 0)$ (iii) $F_1(-\sqrt{34}, 0)$, $F_2(\sqrt{34}, 0)$

(iv) $e = \frac{\sqrt{34}}{3}$ (v) $16\frac{2}{3}$ units

6. (i) 10 units, $4\sqrt{6}$ units (ii) $A(-5, 0)$, $B(5, 0)$ (iii) $F_1(-7, 0)$, $F_2(7, 0)$

(iv) $e = 1\frac{2}{5}$ (v) $9\frac{3}{5}$ units

7. (i) 8 units, 14 units (ii) $A(0, -4)$, $B(0, 4)$ (iii) $F_1(0, -\sqrt{65})$, $F_2(0, \sqrt{65})$

(iv) $e = \frac{\sqrt{65}}{4}$ (v) $24\frac{1}{2}$ units

8. (i) 6 units, $6\sqrt{3}$ units (ii) $A(0, -3)$, $B(0, 3)$ (iii) $F_1(0, -6)$, $F_2(0, 6)$

(iv) $e = 2$ (v) 18 units

9. (i) 12 units, $12\sqrt{3}$ units (ii) $A(0, -6)$, $B(0, 6)$ (iii) $F_1(0, -12)$, $F_2(0, 12)$

(iv) $e = 2$ (v) 36 units

10. (i) $\frac{12}{\sqrt{5}}$ units, 4 units (ii) $A\left(0, -\frac{6}{\sqrt{5}}\right)$, $B\left(0, \frac{6}{\sqrt{5}}\right)$

(iii) $F_1\left(0, -2\sqrt{\frac{14}{5}}\right)$, $F_2\left(0, 2\sqrt{\frac{14}{5}}\right)$ (iv) $e = \frac{\sqrt{14}}{3}$ (v) $\frac{4\sqrt{5}}{3}$ units

11. $\frac{x^2}{36} - \frac{y^2}{28} = 1$ **12.** $\frac{y^2}{25} - \frac{x^2}{39} = 1$ **13.** $\frac{x^2}{25} - \frac{y^2}{4} = 1$

14. $\frac{x^2}{9} - \frac{y^2}{16} = 1$, $e = \frac{5}{3}$ **15.** $\frac{x^2}{25} - \frac{y^2}{20} = 1$ **16.** $\frac{x^2}{4} - \frac{y^2}{12} = 1$

17. $\frac{x^2}{3} - \frac{y^2}{2} = 1$ **18.** $16x^2 - 2y^2 = 1$ **19.** $x^2 - y^2 = 32$

20. $\frac{y^2}{9} - \frac{x^2}{7} = 1$, $(0, \pm 4)$ **21.** $\frac{y^2}{25} - \frac{x^2}{144} = 1$

22. $\frac{y^2}{64} - \frac{x^2}{36} = 1$ **23.** $y^2 - x^2 = 7$

10. Given equation is $\frac{y^2}{(36/5)} - \frac{x^2}{4} = 1.$

$\therefore \quad a^2 = \frac{36}{5}, b^2 = 4$ and $c^2 = (a^2 + b^2) = \left(\frac{36}{5} + 4\right) = \frac{56}{5}.$

Also, $e = \frac{c}{a} = \left(\frac{\sqrt{56}}{\sqrt{5}} \times \frac{\sqrt{5}}{6}\right) = \frac{2\sqrt{14}}{6} = \frac{\sqrt{14}}{3}.$

18. $e = 3 \Leftrightarrow \frac{c}{a} = 3 \Leftrightarrow c = 3a \Leftrightarrow c^2 = 9a^2 \Leftrightarrow a^2 + b^2 = 9a^2 \Leftrightarrow 8a^2 = b^2.$ … (i)

$\frac{2b^2}{a} = 4 \Leftrightarrow b^2 = 2a.$ … (ii)

$\therefore \quad 8a^2 = 2a \Leftrightarrow 2a(4a - 1) = 0 \Leftrightarrow a = \frac{1}{4}.$

$\therefore \quad a^2 = \frac{1}{16}$ and $b^2 = \left(2 \times \frac{1}{4}\right) = \frac{1}{2}.$

19. $2c = 16 \Leftrightarrow c = 8.$ Also, $e = \sqrt{2}.$

Now, $e = \frac{c}{a} \Rightarrow a = \frac{c}{e} = \frac{8}{\sqrt{2}} = 4\sqrt{2}.$

$\therefore \quad b^2 = (c^2 - a^2) = (64 - 32) = 32.$

Thus, $a^2 = 32$ and $b^2 = 32.$

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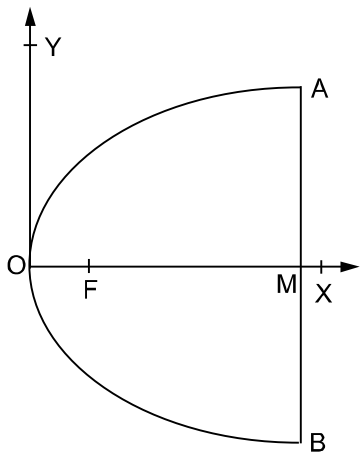
Applications of Conic Sections

Conic sections have their applications in various fields. Some of them are listed below.

1. We know that when an object is thrown in space then the path traced by the object (called a projectile) is a parabola. By using various properties of a parabola, we can find the equation of the path traced by the projectile. Also, we can obtain other useful information such as the greatest height attained by the projectile, its horizontal range, and so on.
2. In a parabolic reflector, light rays or sound waves directed parallel to its axis converge at the focus and are then reflected by it, parallel to the axis. Because of this property, parabolic reflectors are used in cars, automobiles, loudspeakers, solar cookers, telescopes, etc.
3. Cables shaped as parabolic arcs are used in the construction of suspension bridges.
4. The planets of the solar system move in elliptical paths with the sun at one of the foci. Artificial satellites are made to move in elliptical paths around the earth.
5. Hyperbolas have their applications in the field of ballistics. Suppose a gun is fired and its sound reaches two posts, situated at the two foci of a hyperbola, at different times. From the time difference, the distance between the two listening posts can be calculated.

SOLVED EXAMPLES

EXAMPLE 1 *The focus of a parabolic mirror is at a distance of* 5 cm *from its vertex and the mirror is* 15 cm *deep. Find the diameter of the mirror.*



SOLUTION Let us take the vertex of the mirror at the origin with its axis lying along the positive direction of the x-axis.

Let the equation of the parabolic shape of the mirror be $y^2 = 4ax$.

Since the distance between the focus and the vertex is 5 cm, we have $a = 5$.

$\therefore$ the equation of the parabola is

$$y^2 = 4 \times 5 \times x, \text{ i.e., } y^2 = 20x. \qquad \ldots \text{(i)}$$

The depth of the mirror, $OM = 15$ cm.

$\therefore$ the coordinates of one end of the diameter AB are $(15, p)$, where $AM = p$ cm.

Clearly, $A(15, p)$ lies on the parabola $y^2 = 20x$.

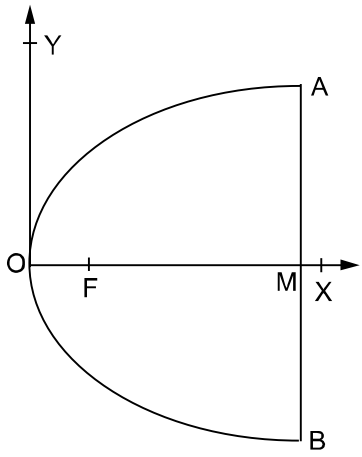
$\therefore \quad p^2 = (20 \times 15) \quad \Rightarrow \quad p = \sqrt{300} = 10\sqrt{3}$

$\Rightarrow \quad AM = 10\sqrt{3}$ cm $\quad \Rightarrow \quad AB = 2 \times AM = 20\sqrt{3}$ cm.

Hence, the diameter of the mirror is $20\sqrt{3}$ cm.

EXAMPLE 2 *A parabolic reflector is* 9 cm *deep and its diameter is* 24 cm. *How far is its focus from the vertex?*

SOLUTION Let us take the vertex of the reflector at the origin, and let its axis lie along the positive direction of the x-axis.



Let the equation of the parabolic shape of the reflector be

$$y^2 = 4ax. \qquad \ldots \text{(i)}$$

As given, we have

$$OM = 9 \text{ cm and } AB = 24 \text{ cm}$$

$$\Rightarrow \quad OM = 9 \text{ cm and } AM = \frac{1}{2} AB = 12 \text{ cm}.$$

$\therefore$ the coordinates of A are $(9, 12)$.

Since A lies on (i), we have

$$12 \times 12 = 4 \times a \times 9 \Rightarrow a = 4.$$

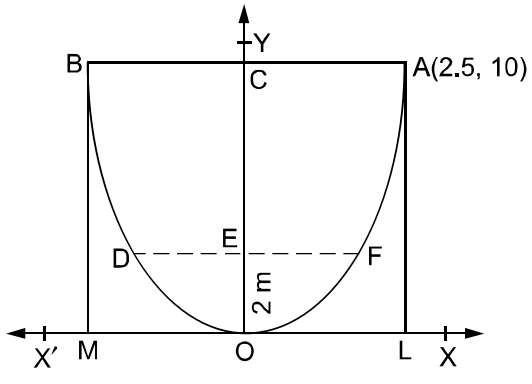
$\therefore$ the equation of the parabola is $y^2 = 16x$.

Its focus is $F(a, 0)$, i.e., $F(4, 0)$.

Hence, the focus is at a distance of 4 cm from the vertex.

EXAMPLE 3 *An arch is in the form of a parabola with its axis vertical. The arch is* 10 m *high and* 5 m *wide at the base. Find its width at a distance of* 2 m *from its vertex.*

SOLUTION Let O be the origin and OY be the vertical axis of the parabolic arch.



Then, the equation of the arch is

$$x^2 = 4ay. \qquad \ldots \text{(i)}$$

$\therefore BA = 5$ m and $OC = 10$ m.

Now, $OL = \frac{1}{2} ML = \frac{1}{2} BA = 2.5$ m.

And, $AL = OC = 10$ m.

$\therefore$ the coordinates of A are $(2.5, 10)$. Since $A(2.5, 10)$ lies on (i), we have

$$(2.5)^2 = 4a \times 10 \quad \Rightarrow \quad a = \left(\frac{2.5 \times 2.5}{4 \times 10}\right) = \frac{5}{32}.$$

$$\therefore \quad x^2 = \left(4 \times \frac{5}{32} \times y\right) \Rightarrow \quad x^2 = \frac{5}{8}y. \qquad \ldots \text{(ii)}$$

Draw $DEF \parallel ML$ such that $OE = 2$ m.

Let the coordinates of F be $F(x_1, 2)$.

Since $F(x_1, 2)$ lies on (ii), we have

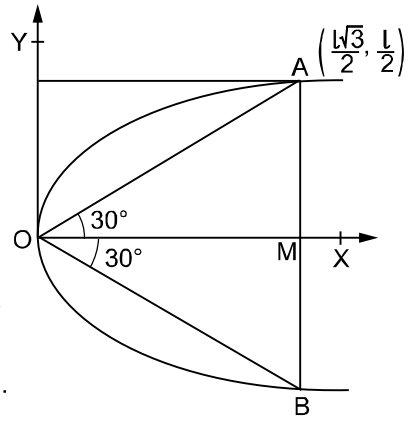
$$x_1^2 = \left(\frac{5}{8} \times 2\right) \Rightarrow \quad x_1^2 = \frac{5}{4} \quad \Rightarrow \quad x_1 = \frac{\sqrt{5}}{2}.$$

$$\therefore \quad EF = \frac{\sqrt{5}}{2} \text{ m} \quad \Rightarrow \quad DF = 2 \times EF = \left(2 \times \frac{\sqrt{5}}{2}\right) \text{m} = \sqrt{5} \text{ m}.$$

Hence, the required width of the arch at a distance of 2 m from its vertex is $\sqrt{5}$ metres.

EXAMPLE 4 *An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ so that one angular point of the triangle is at the vertex of the parabola. Find the length of each side of the triangle.*

SOLUTION Let $\triangle OAB$ be an equilateral triangle inscribed in the parabola $y^2 = 4ax$.



Let $OA = OB = AB = l$.

Let AB cut the x-axis at M.

Then, $\angle AOM = \angle BOM = 30°$.

$$\therefore \frac{OM}{OA} = \cos 30° \Rightarrow OM = l \cos 30° = \frac{l\sqrt{3}}{2},$$

and $$\frac{AM}{OA} = \sin 30° \Rightarrow AM = l \sin 30° = \frac{l}{2}.$$

$\therefore$ the coordinates of A are $\left(\frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$.

Since A lies on the parabola $y^2 = 4ax$, we have

$$\frac{l^2}{4} = 4a \times \frac{l\sqrt{3}}{2} \quad \Rightarrow \quad l = 8a\sqrt{3}.$$

Hence, the length of each side of the triangle is $8a\sqrt{3}$ units.

EXAMPLE 5 *Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.*

SOLUTION The vertex of the parabola $x^2 = 12y$ is $O(0, 0)$.

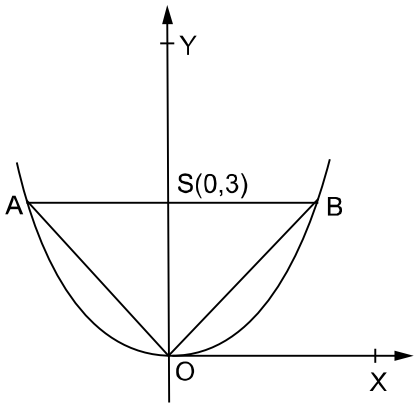
Comparing $x^2 = 12y$ with $x^2 = 4ay$, we get $a = 3$.

The coordinates of its focus S are $(0, 3)$.

Clearly, the ends of its latus rectum are

$A(-2a, a)$ and $B(2a, a)$

i.e., $A(-6, 3)$ and $B(6, 3)$.



$$\therefore \text{ area of } \triangle OBA = \frac{1}{2} \cdot \begin{vmatrix} 0 & 0 & 1 \\ 6 & 3 & 1 \\ -6 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{1 \times (18 + 18)\}$$

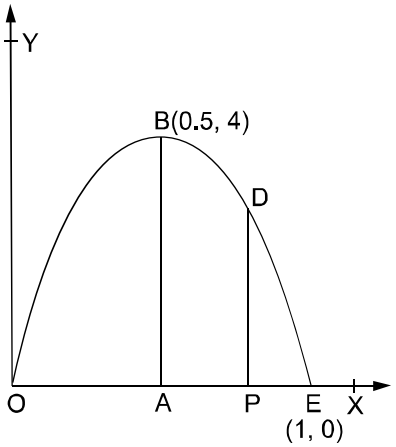
$$= 18 \text{ sq units.}$$

EXAMPLE 6 *A water jet coming out of the small opening O of a fountain reaches its maximum height of* 4 *metres at a distance of* 0.5 *metre from the vertical. Find the height of the jet above the horizontal OX at a distance of* 0.75 *metre from the point O.*

SOLUTION The path of a water jet is a parabola.

Let the equation of this parabolic path be

$$y = ax^2 + bx + c. \qquad \ldots \text{(i)}$$



Let AB be the maximum height reached.

The path is symmetrical about AB.

Let the jet strike the x-axis at E.

Then, $AE = OA = 0.5$ m

$\Rightarrow \quad OE = 2(OA) = (2 \times 0.5)$ m $= 1$ m.

Clearly, $AB = 4$ m.

Thus, the points O, B and E are

$O(0, 0)$, $B(0.5, 4)$ and $E(1, 0)$.

Since these points lie on (i), we have

$$c = 0, \quad \frac{1}{4}a + \frac{1}{2}b = 4 \quad \text{and} \quad a + b = 0$$

$$\Rightarrow \quad c = 0, \quad a + 2b = 16 \quad \text{and} \quad a + b = 0$$

$$\Rightarrow \quad a = -16, \quad b = 16 \quad \text{and} \quad c = 0.$$

$\therefore$ the equation of the parabola is $y = -16x^2 + 16x$. $\qquad \ldots \text{(ii)}$

Let P be a point on OE such that $OP = 0.75$ m.

Draw $PD \perp OX$, meeting the parabola at D.

Let $PD = h$ metres.

Then, the coordinates of D are $(0.75, h)$.

Since $D(0.75, h)$ lies on (ii), we have

$$h = -16 \times \left(\frac{3}{4}\right)^2 + \left(16 \times \frac{3}{4}\right) \quad \Rightarrow \quad h = (-9 + 12) = 3.$$

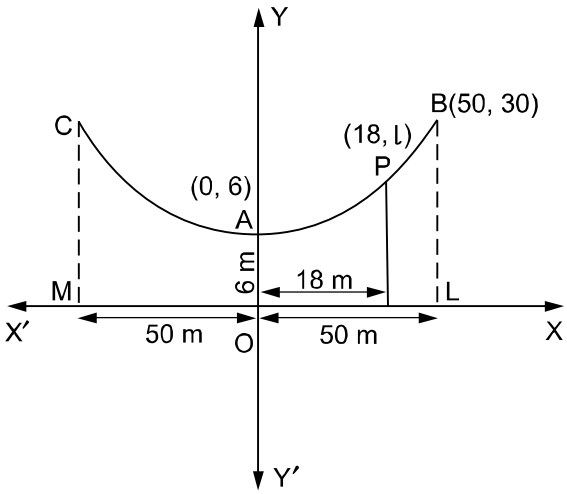
$\therefore$ the required height, $PD = h$ m $= 3$ m.

EXAMPLE 7 *The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and* 100 m *long is supported by vertical wires attached to the cable, the longest wire being* 30 m *and the shortest being* 6 m. *Find the length of a supporting wire attached to the roadway* 18 m *from the middle.*

SOLUTION Let $X'OX$ be the x-axis, taken along the roadway, and let YOY' be the y-axis.

Let CAB be the bridge in the form of a parabola having its vertex at $A(0, 6)$ such that A is the centre of the bridge.

Let BL and CM be the longest vertical wires, each of length 30 m, and let OA be the shortest vertical wire, 6 m in length.



The equation of the parabolic form of the bridge is

$$x^2 = 4a(y - 6). \qquad \ldots \text{(i)}$$

It passes through $B(50, 30)$.

Putting $x = 50$ and $y = 30$ in (i), we get

$$2500 = 4a(30 - 6) \quad \Rightarrow \quad a = \frac{2500}{96}.$$

Putting $a = \dfrac{2500}{96}$ in (i), we get

$$x^2 = 4 \times \frac{2500}{96}(y - 6) \quad \Rightarrow \quad 6x^2 = 625y - 3750. \qquad \ldots \text{(ii)}$$

Let l m be the length of the supporting wire 18 m from the middle.

Then, $P(18, l)$ must satisfy (ii).

$$\therefore \quad 6 \times 18 \times 18 = 625l - 3750 \quad \Rightarrow \quad 625l = 5694$$
$$\Rightarrow \quad l = 9.11.$$

Hence, the length of the supporting wire 18 m from the middle is 9.11 m.

EXAMPLE 8 *A rod of length* 12 cm *moves with its ends always touching the coordinate axes. Determine the equation of the path of a moving point P on the rod which is* 3 cm *from the end in contact with the x-axis.*

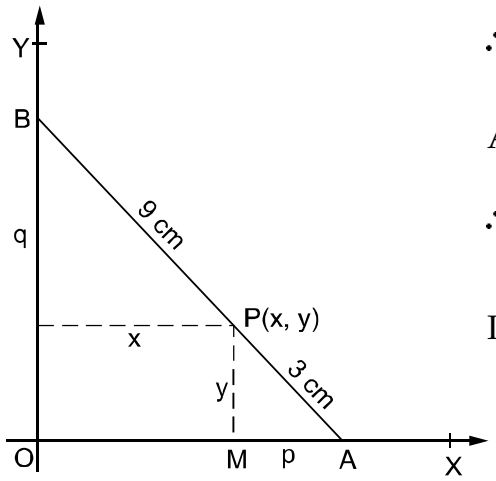
SOLUTION Let AB be the rod and $P(x, y)$ be a point on it such that

$$AP = 3 \text{ cm} \quad \text{and} \quad PB = 9 \text{ cm}.$$

From P, draw $PM \perp OX$ and $PN \perp OY$.

Let $AM = p$ and $BN = q$.

Then $\triangle BNP$ and $\triangle PMA$ are similar.



$$\therefore \quad \frac{BN}{PM} = \frac{BP}{PA} \Rightarrow \frac{q}{y} = \frac{9}{3} \Rightarrow q = 3y.$$

$$\text{And, } \frac{MA}{NP} = \frac{PA}{BP} \Rightarrow \frac{p}{x} = \frac{3}{9} \Rightarrow p = \frac{1}{3}x.$$

$$\therefore \quad OA = OM + MA = x + p = x + \frac{1}{3}x = \frac{4x}{3},$$

$$\text{and } OB = ON + BN = y + q = y + 3y = 4y.$$

In $\triangle BOA$, we have

$$OA^2 + OB^2 = AB^2$$

$$\Rightarrow \left(\frac{4x}{3}\right)^2 + (4y)^2 = (12)^2$$

$$\Rightarrow \frac{16x^2}{9} + 16y^2 = 144$$

$$\Rightarrow \frac{x^2}{81} + \frac{y^2}{9} = 1.$$

Hence, the path of P is an ellipse whose equation is $\frac{x^2}{81} + \frac{y^2}{9} = 1.$

EXAMPLE 9 *A man running in a racecourse notes that the sum of the distances of the two flag posts from him is always* 10 m, *and the distance between the flag posts is* 8 m. *Find the equation of the path traced by the man.*

SOLUTION We know that an ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant.

So, the path traced by the man is an ellipse.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2).$$

Clearly, $\quad 2a = 10 \quad \text{and} \quad 2ae = 8$

$$\Rightarrow \quad a = 5 \quad \text{and} \quad e = \frac{4}{5}$$

$$\Rightarrow \quad b^2 = a^2(1 - e^2) = 25\left(1 - \frac{16}{25}\right) = 9$$

$$\Rightarrow \quad b = 3.$$

Hence, the equation of the path is $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

EXAMPLE 10 *An arch is in the form of a semi-ellipse. It is* 8 m *wide and* 2 m *high at the centre. Find the height of the arch at a point* 1.5 m *from one end.*

SOLUTION Let the equation of the ellipse of which the given arch is a part be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \ldots \text{(i)}$$

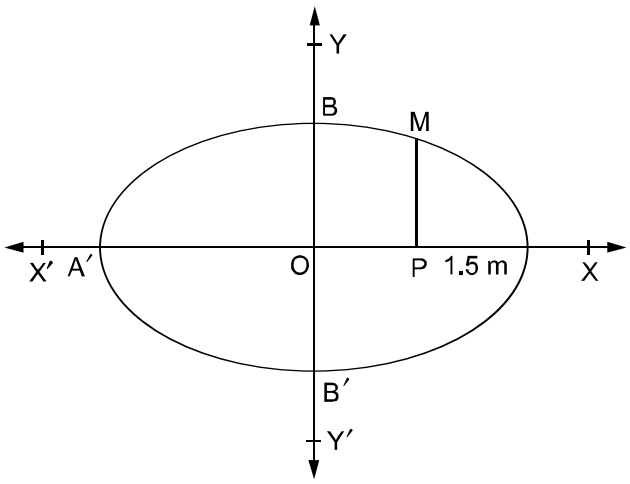
It is being given that

$$A'A = 8\,\text{m} \quad \text{and} \quad OB = 2\,\text{m}$$
$$\Rightarrow \quad 2a = 8 \quad \text{and} \quad b = 2$$
$$\Rightarrow \quad a = 4 \quad \text{and} \quad b = 2.$$

So, the equation of the arch is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

Let P be a point on $A'OA$ such that $AP = 1.5$ m.



Draw $PM \perp A'A$, meeting the ellipse at M.

Then, $AP = 1.5$ m (given).

Now, $OP = (OA - AP) = (4 - 1.5)$ m $= 2.5$ m. Let $PM = p$ m.

So, the coordinates of M are $(2.5, p)$.

Since M lies on the ellipse, we have

$$\frac{(2.5)^2}{16} + \frac{p^2}{4} = 1 \;\Rightarrow\; \frac{p^2}{4} = \left(1 - \frac{625}{1600}\right) \;\Rightarrow\; p^2 = \frac{39}{16}$$

$$\Rightarrow\; p = \frac{\sqrt{39}}{4} = \frac{6.24}{4} = 1.56.$$

Hence, the height of the arch at a point 1.5 m from one end is 1.56 m.

EXERCISE 25

1. The focus of a parabolic mirror is at a distance of 6 cm from its vertex. If the mirror is 20 cm deep, find its diameter.
2. A parabolic reflector is 5 cm deep and its diameter is 20 cm. How far is its focus from the vertex?
3. The towers of a bridge, hung in the form of a parabola, have their tops 30 m above the roadway, and are 200 m apart. If the cable is 5 m above the roadway at the centre of the bridge, find the length of the vertical supporting cable, 30 m from the centre.

4. A rod of length 15 cm moves with its ends always touching the coordinate axes. Find the equation of the locus of a point P on the rod, which is at a distance of 3 cm from the end in contact with the x-axis.

5. A beam is supported at its ends by supports which are 12 m apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre, and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

ANSWERS (EXERCISE 25)

1. $8\sqrt{30}$ cm **2.** 5 cm **3.** 7.25 m **4.** $5x^2 + 9y^2 = 81$ **5.** $2\sqrt{6}$ m

HINTS TO SOME SELECTED QUESTIONS

5. The beam takes the shape of a parabola, whose equation is of the form $x^2 = 4ay$. … (i)

Since the point $P\left(6, \frac{3}{100}\right)$ lies on it, we have

$$\left(4 \times a \times \frac{3}{100}\right) = 36 \Leftrightarrow a = 300 \text{ m}.$$

Let AB be the deflection. Then,

$AB = \frac{1}{100}$ m and B is $B\left(x, \frac{2}{100}\right)$.

$$\therefore \quad x^2 = 4 \times 300 \times \frac{2}{100} \Leftrightarrow x = 2\sqrt{6} \text{ m}.$$